

Renormalization of the Mass Gap

V. Gogokhia

Received: 17 July 2009 / Accepted: 15 September 2009 / Published online: 29 September 2009
© Springer Science+Business Media, LLC 2009

Abstract The full gluon propagator relevant for the description of the truly non-perturbative QCD dynamics, the so-called intrinsically non-perturbative gluon propagator has been derived in our previous work. It explicitly depends on the regularized mass gap, which dominates its structure at small gluon momentum. It is automatically transversal in a gauge invariant way. It is characterized by the presence of severe infrared singularities at small gluon momentum, so the gluons remain massless, and this does not depend on the gauge choice. In this paper we have shown how precisely the renormalization program for the regularized mass gap should be performed. We have also shown how precisely severe infrared singularities should be correctly treated. This allowed to analytically formulate the exact and gauge-invariant criteria of gluon and quark confinement. After the renormalization program is completed, one can derive the gluon propagator applicable for the calculation of physical observables, processes, etc., in low-energy QCD from first principles.

Keywords QCD · Gluon propagator · Infrared singularities · Generalized functions · Renormalization · Mass Gap · Quark and Gluon criteria of confinement

1 Introduction

In our previous work [1] the mass gap responsible for the non-perturbative dynamics in QCD has been introduced through the definition of the subtracted full gluon self-energy. So it is defined as the value of the regularized full gluon self-energy at some finite point. The mass gap is mainly generated by the nonlinear interaction of massless gluon modes. We have explicitly shown that QCD is the color gauge invariant theory at non-zero mass gap as well. All this allows one to establish the structure of the full gluon propagator in the explicit presence of the mass gap [1]. In this case, the two independent general types of the formal solutions for the regularized full gluon propagator have been found [2]. No truncations/approximations/assumptions, as well as no special gauge choice are made for the

V. Gogokhia (✉)
Depart. Theor. Phys., HAS, CRIP, RMKI, P.O.B. 49, 1525, Budapest 114, Hungary
e-mail: gogohia@rmki.kfki.hu

regularized skeleton loop integrals, contributing to the full gluon self-energy. The so-called massive solution, which leads to an effective gluon mass has been established. The general nonlinear iteration solution for the full gluon propagator depends explicitly on the mass gap. It is always severely singular in the $q^2 \rightarrow 0$ limit, so the gluons remain massless, and this does not depend on the gauge choice. However, we have argued that only its intrinsically non-perturbative (INP) part, defined by the subtraction of all the types of the perturbative theory (PT) contributions (“contaminations”) from the general nonlinear iteration solution, is interesting for confinement. In this way it becomes automatically transversal in a gauge invariant way and excludes the free gluons from the theory. This is important for correct understanding of the color confinement mechanism, as underlined in our previous works [1, 2].

The INP gluon propagator is to be used for the numerical calculations of physical observables, processes, etc. in low-energy QCD from first principles. But before the two important problems remain to solve. The first problem is how to perform the renormalization program for the regularized mass gap $\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$, and to see whether the mass gap survives it or not. The second problem is how to treat correctly severe infrared (IR) singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, \dots$ inevitably present in this solution (let us remind that the signature is Euclidean). Just these problems are to be addressed and solved in this paper.

2 The INP Gluon Propagator

The full gluon propagator, which is relevant for the description of the truly NP QCD dynamics, the so-called INP gluon propagator derived in Ref. [2] is as follows:

$$D_{\mu\nu}^{INP}(q) = iT_{\mu\nu}(q)d^{INP}(q^2)\frac{1}{q^2} = iT_{\mu\nu}(q)\frac{\Delta^2}{(q^2)^2}L(q^2), \tag{2.1}$$

and

$$L(q^2) \equiv L(q^2; \Delta^2) = \sum_{k=0}^{\infty} \left(\frac{\Delta^2}{q^2}\right)^k \Phi_k = \sum_{k=0}^{\infty} \left(\frac{\Delta^2}{q^2}\right)^k \sum_{m=0}^{\infty} \Phi_{km}, \tag{2.2}$$

where $\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$ and the residues $\Phi_k \equiv \Phi_k(\lambda, \alpha, \xi, g^2)$ and thus $\Phi_{km} \equiv \Phi_{km}(\lambda, \alpha, \xi, g^2)$ as well.

Let us recall some interesting features of the INP gluon propagator (2.1) [2]. First of all, it depends only on the transversal degrees of freedom of gauge bosons. Also, its functional dependence is uniquely fixed up to the expressions for the residues Φ_k , and it is valid in the whole energy/momentum range. It explicitly depends on the mass gap, so that when it formally goes to zero (the formal PT $\Delta^2 = 0$ limit) this solution vanishes. Also, it is characterized by the explicit presence of severe (i.e., NP) IR singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, \dots$ only. The sum over m indicates that an infinite number of iterations (all iterations) invokes each severe IR singularity labeled by k . Apart from the structure $(\Delta^2/(q^2)^2)$ it is nothing but the corresponding Laurent expansion. This solution is completely free of all the types of the PT contributions (“contaminations”), and thus is exactly and uniquely separated from the PT gluon propagator, indeed. Due to the regular dependence on the mass gap, it dominates over the free gluon propagator in the deep IR ($q^2 \rightarrow 0$) limit and it is suppressed in the deep ultraviolet (UV) ($q^2 \rightarrow \infty$) limit (let us remind that the free gluon propagator behaves like $1/q^2$ in the whole momentum range). In the rest of this paper we will omit the superscript “INP”, for simplicity. We will restore it if it will be necessary for clarity of our discussion.

3 The General Multiplicative Renormalization Program

One of the remarkable features of the solution (2.1)–(2.2) is that its asymptotic at infinity ($q^2 \rightarrow \infty$) is to be determined by its $\Delta^2/(q^2)^2$ structure only, since all other terms in its expansion are suppressed in this limit. It is well known that such a behavior at infinity is not dangerous for the renormalizability of QCD [3–6]. However, the regularized mass gap itself is of the NP origin. So its renormalization in order to get finite result in the $\lambda \rightarrow \infty$ limit cannot be done in standard way. Another main problem, closely connected to the mass gap, is its violent structure in the deep IR region ($q^2 \rightarrow 0$). Fortunately, there exist two mathematical theories which are of a great help in this case. The distribution theory (DT) [7], which should be complemented by the dimensional regularization method (DRM) [8] (some important issues of these theories are present and discussed in Appendix A in more detail).

For further purpose, let us present our solution (2.1) as follows:

$$D_{\mu\nu}(q) = iT_{\mu\nu}(q) \sum_{k=0}^{\infty} (\Delta^2)^{k+1} (q^2)^{-2-k} \Phi_k. \tag{3.1}$$

As it has been emphasized in Appendix A, a special regularization expansion is to be used in order to deal with the severe IR singularities $(q^2)^{-2-k}$, since standard methods fail to control them. If q^2 is an independent loop variable, then the corresponding dimensionally regularized expansion is given in (8.10), namely

$$(q^2)^{-2-k} = \frac{1}{\epsilon} \left[a(k) [\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+. \tag{3.2}$$

Substituting it into the previous expansion, one obtains

$$D_{\mu\nu}(q) = iT_{\mu\nu}(q) \frac{1}{\epsilon} \sum_{k=0}^{\infty} (\Delta^2)^{k+1} \Phi_k \left[a(k) [\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+, \tag{3.3}$$

so instead of the Laurent expansion (2.1) we obtain dimensionally regularized (in powers of ϵ) expansion for the INP gluon propagator. In the presence of the IR regularization parameter ϵ all the quantities and parameters depend, in general, on it. For further purpose it is convenient to introduce the following short-hand notations, namely $\Delta^2 \equiv \Delta^2(\lambda, \epsilon)$ and $\Phi_k \equiv \Phi_k(\lambda, \epsilon)$ and omit the dependence on the other parameters, for simplicity. Evidently, the renormalization of the mass gap only is what that matters, since $\Phi_k(\lambda, \epsilon)$ is the dimensionless quantity (see below).

So let us define the UV and IR renormalized mass gap as follows:

$$\Delta^2(\lambda; \epsilon) = Z(\lambda; \epsilon) \bar{\Delta}_R^2, \quad \lambda \rightarrow \infty, \quad \epsilon \rightarrow 0^+, \tag{3.4}$$

where $Z \equiv Z(\lambda; \epsilon)$ is the multiplicative renormalization (MR) constant of the mass gap. Here and everywhere below all the quantities with sub- or superscript “R” denote the quantities which exist in the $\lambda \rightarrow \infty$ limit, i.e., by definition, they are UV renormalized (and their dependence on other parameters is to be neglected, apart from the coupling constant, which can be “running” itself, see Sect. 5 below). At the same time, all the quantities with bar are IR renormalized, i.e., by definition, they exist as $\epsilon \rightarrow 0^+$. That is why here and everywhere below in all the renormalized quantities the dependence on λ and ϵ is not shown explicitly, as well as on other possible parameters. The factorization of the corresponding

MR constants in order to get the finite results separately in the $\lambda \rightarrow \infty$ and $\epsilon \rightarrow 0^+$ limits is a particular case of the general MR program in order to remove the dependence on λ and ϵ in their corresponding limits. The NP (quadratic) UV divergences present in the regularized mass gap are in close intrinsic link with the NP IR singularities, which both are present in the INP QCD gluon propagator (2.1).

Substituting the relation (3.4) into the previous expansion (3.3), one arrives at

$$D_{\mu\nu}(q) = iT_{\mu\nu}(q) \frac{1}{\epsilon} \sum_{k=0}^{\infty} (\bar{\Delta}_R^2)^{k+1} Z^{k+1}(\lambda; \epsilon) \Phi_k(\lambda; \epsilon) \left[a(k)[\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+. \tag{3.5}$$

The only way to remove the pole $1/\epsilon$ from the previous expansion and get the final results after the removal the dependence on λ and ϵ is to put

$$Z^{k+1}(\lambda; \epsilon) \Phi_k(\lambda; \epsilon) = \epsilon \bar{\Omega}_k^R, \quad \epsilon \rightarrow 0^+, \lambda \rightarrow \infty, k = 0, 1, 2, 3, \dots, \tag{3.6}$$

since all the NP IR singularities labeled by k are independent from each other. This is the general convergence condition which makes the gluon propagator free of λ and ϵ in their corresponding limits.

So the previous expansion becomes

$$\bar{D}_{\mu\nu}^R(q) = iT_{\mu\nu}(q) \sum_{k=0}^{\infty} (\bar{\Delta}_R^2)^{k+1} \bar{\Omega}_k^R \left[a(k)[\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+. \tag{3.7}$$

Through the chain of the relations $\bar{\Omega}_k^R = [\bar{\Psi}_k^R]^{k+1}$, $\bar{\Delta}_R^2 \bar{\Psi}_k^R = (\bar{\Delta}_R^2)_k$, one obtains

$$\bar{D}_{\mu\nu}^R(q) = iT_{\mu\nu}(q) \sum_{k=0}^{\infty} [(\bar{\Delta}_R^2)_k]^{k+1} \left[a(k)[\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+, \tag{3.8}$$

which shows that one can start from the k -dependent regularized mass gap in (3.3), but, nevertheless, coming to the same final (3.8), containing the k -dependent renormalized mass gap.

The renormalization of the mass gap itself has been defined by the relation (3.4). Its MR constant Z remains undetermined, but this is not already the problem, since the renormalized gluon propagator (3.7) depends on the renormalized mass gap $\bar{\Delta}_R^2$. So we consider it as the physical mass gap within our approach. Precisely this quantity should be positive, finite, gauge- and α -independent, etc., it should exist when $\lambda \rightarrow \infty$ and $\epsilon \rightarrow 0^+$. The renormalization of the mass gap is an example of the NP MR program. We were able to accumulate all the quadratic divergences into the renormalization of the mass gap. Due to the renormalization the quadratic divergences parameterized as the regularized mass gap Δ^2 may be absorbed in its re-definition leading thus finally to the physical mass gap.

Concluding, since all the parameters in the expansion (3.7) are expressed in the IR renormalized quantities, we can go to the $\epsilon \rightarrow 0^+$ limit, without encountering any problems in this limit now. Also, we have found of no practical use to introduce the MR constant Z_3 for the gluon propagator itself separately from that of the mass gap (3.4).

3.1 The General Criterion of Gluon Confinement

We are now in the position to analytically formulate the general criterion of gluon confinement. It will be instructive to analytically formulate the general criterion of quark confinement as well. Let us begin with the former one.

The dimensionally renormalized (within the DT complemented by the DRM) expression for the relevant gluon propagator in INP QCD is (3.7). Substituting back into this equation the expansion (3.2), one obtains

$$D_{\mu\nu}^R(q) = \epsilon \times iT_{\mu\nu}(q) \sum_{k=0}^{\infty} (\bar{\Delta}_R^2)^{k+1} \bar{\Omega}_k^R(q^2)^{-2-k}, \quad \epsilon \rightarrow 0^+. \tag{3.9}$$

Let us underline that this is the general expression for the renormalized gluon propagator, since it does not depend on whether the gluon momentum is independent skeleton loop variable or not. Due to the distribution nature of the NP IR singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3 \dots$ (see Appendix A as well), the two principally different cases should be separately considered.

1. If the gluon momentum q is an independent skeleton loop variable, then, as emphasized repeatedly above, the initial $(q^2)^{-2-k}$ NP IR singularities should be regularized with the help of the expansion (3.2). Finally one arrives at (3.7) in the $\epsilon \rightarrow 0^+$ limit, as it should be.

Let us note in advance that beyond the one-loop skeleton integrals the analysis should be done in a more sophisticated way, otherwise the appearance of the product of at least two δ functions at the same point is possible. However this product is not defined in the DT [7]. So in the multi-loop skeleton diagrams instead of the δ functions in the residues their derivatives may appear (see Ref. [7] and Appendix A in this paper). They should be treated in the sense of the DT. Fortunately, as mentioned in Appendix A, the IR renormalization of the theory is not undermined, since a pole in ϵ is always a simple pole $1/\epsilon$ for each independent skeleton loop variable (see the dimensionally regularized expansion (8.10)). That is why the starting expression for the relevant gluon propagator in INP QCD is always the general expression (3.9). It makes it possible to perform formal algebraic operations (first of all multiplication) on the corresponding Laurent expansions. The final product is again the Laurent expansion. This allows one to use the dimensionally regularized expansion (8.10), and thus to avoid the multiplication of the δ -functions at the same point, if the starting expansion will be expansion (3.7) for each gluon propagator.

2. The necessary and sufficient conditions for gluon confinement are:

(i). If, however, the gluon momentum q is not a loop variable (i.e., it is external momentum), then the initial $(q^2)^{-2-k}$ NP IR singularities cannot be treated as distributions, i.e., the regularization expansion (3.2) is not the case to be used. The functions $(q^2)^{-2-k}$ are the standard ones, and the relevant gluon propagator (3.9) vanishes as ϵ goes to zero, i.e.,

$$D_{\mu\nu}^R(q) = \epsilon \times iT_{\mu\nu}(q) \sum_{k=0}^{\infty} [(\bar{\Delta}_R^2)_k]^{k+1} (q^2)^{-2-k} \sim \epsilon, \quad \epsilon \rightarrow 0^+. \tag{3.10}$$

It is worth emphasizing that the final $\epsilon \rightarrow 0^+$ limit is permitted to take only after expressing all the Green's functions, parameters and mass gap in terms of their IR renormalized counterparts because they, by definition, exist in this limit. This behavior is gauge-invariant, does not depend on any truncations/approximations, etc., and thus it is a general one. It prevents the transversal dressed gluons to appear in asymptotic states, so color dressed gluons can never be isolated. This is the *first necessary* condition for gluon confinement.

(ii). The *second sufficient* condition of gluon confinement is the absence of the free gluons in the corresponding theory. Just such a theory has been formulated in our previous works [1, 2], namely INP QCD. Let us remind that its full gluon propagator (2.1) has no free gluon propagator limit when the interaction is to be switched off ($\Delta^2 = 0$). We argued that

it should be used for the calculations of physical observables, processes in low-energy QCD from first principles. We consider both the suppression of the colored dressed gluons at large distances and the absence of the free colored gluons in the theory as the exact criterion of gluon confinement (for its initial formulation see Ref. [9]).

3.2 The General Criterion of Quark Confinement

It is instructive to formulate here the quark confinement criterion in advance as well. It consists of the two independent conditions.

1. *The first necessary condition*, formulated at the fundamental (microscopic) quark-gluon level, is the absence of the pole-type singularities in the quark Green's function at any gauge on the real axe at some finite point in the complex momentum plane, i.e.,

$$S(p) \neq \frac{Z_2}{\hat{p} - m_{ph}}, \quad (3.11)$$

where Z_2 is the standard quark wave function renormalization constant, while m_{ph} is the mass to which a physical meaning could be assigned. In other words, the quark always remains an off-mass-shell object. Such an understanding (interpretation) of quark confinement comes apparently from the Gribov's approach to quark confinement [10] and Preparata's massive quark model (MQM) in which external quark legs were approximated by entire functions [11]. A quark propagator may or may not be an entire function, but in any case the pole of the first order (like the electron propagator has in QED) should disappear (see, for example Refs. [12–14] and references therein).

2. *The second sufficient condition*, formulated at the hadronic (macroscopic) level, is the existence of the discrete spectrum only (no continuum) in bound-states, in order to prevent quarks to appear in asymptotic states. This condition comes apparently from the 't Hooft's model for two-dimensional QCD with N_c large limit [15] (see also Refs. [12, 16]).

This definition of quark confinement in the momentum space is gauge invariant, flavor independent, i.e., valid for all types of quarks (light or heavy), etc., and thus it is a general one. The Wilson criterion of quark confinement formulated in the configuration space—Area law [17, 18] is relevant only for heavy quarks, as well as a linear rising potential between static (heavy) quarks [19], also “seen” by lattice QCD [20, 21].

At nonzero temperature and density, for example in the quark-gluon plasma (QGP) [22, 23] (and references therein), the bound-states will be dissolved, so the second sufficient condition does not work any more. However, the first necessary condition remains always valid, of course. In other words, by increasing temperature or density there is no way to put quarks on the mass-shell. So that what is known as the De-confinement phase transition in QGP is in fact the De-hadronization phase transition. De-confinement is about the liberation of the colored objects from the vacuum and not from the bound-states. In the QCD ground-state there are many colored objects such as quarks, gluons, instantons and may be something else. Since color confinement is absolute and permanent, none of these colored objects can appear in physical spectrum, and thus De-confinement phase transition does not exist, in principle.

4 Physical Limits

We introduced the renormalized mass gap in the relation (3.4), defined its existence when the dimensionless UV regulating parameter λ goes to infinity, i.e., in the $\lambda \rightarrow \infty$ limit.

However, nothing was said about the behavior of the coupling constant squared g^2 in this limit. In principal, it may also depend on λ , becoming thus the so-called “running” effective charge $g^2 \sim \alpha_s \equiv \alpha_s(\lambda)$. In the general composition (3.4)

$$Z^{-1}(\lambda, \alpha_s(\lambda))\Delta^2(\lambda, \alpha_s(\lambda)), \tag{4.1}$$

all the possible types of the effective charge behavior in the $\lambda \rightarrow \infty$ limit should be considered independently from each other (the dependence on other parameters is not shown explicitly, as unimportant for further discussion).

1). If $\alpha_s(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, then one recovers the strong coupling regime. Evidently, just this finite limit can be defined as the renormalized mass gap (3.4), i.e., in fact

$$Z^{-1}(\lambda, \alpha_s(\lambda))\Delta^2(\lambda, \alpha_s(\lambda)) = \Delta_R^2, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow \infty. \tag{4.2}$$

Apparently, only this mass gap can be identified/related with/to the Jaffe–Witten (JW) mass gap discussed in Ref. [24] (see Sect. 7 below as well).

2). If $\alpha_s(\lambda) \rightarrow c$ as $\lambda \rightarrow \infty$, where c is a finite constant, then it can be put unity, not losing generality. This means that the effective charge becomes unity, and this is only possible for the free gluon propagator. But the free gluon propagator contains none of the mass scale parameters, so in fact

$$Z^{-1}(\lambda, \alpha_s(\lambda))\Delta^2(\lambda, \alpha_s(\lambda)) = 0, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow 1. \tag{4.3}$$

3). If $\alpha_s(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$, then one recovers the weak coupling regime. Evidently, just this finite limit can be defined as Λ_{QCD}^2 , (however, see Sect. 7 below) i.e., in fact

$$Z^{-1}(\lambda, \alpha_s(\lambda))\Delta^2(\lambda, \alpha_s(\lambda)) = \Lambda_{QCD}^2, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow 0. \tag{4.4}$$

There is no doubt that the regularized mass gap may provide the existence of the two different physical mass scale parameters after the renormalization program is performed. Though these two physical parameters show up explicitly at different regimes, nevertheless, numerically they may not be very different, indeed, as emphasized in Ref. [1]. Our mass gap Δ_R^2 determines the power-type deviation of the full gluon propagator from the free one in the $q^2 \rightarrow 0$ limit. The region of small q^2 is interesting for all the NP effects in QCD. This once more emphasizes the close link between the behavior of QCD at large distances and its INP dynamics. At the same time, the asymptotic QCD scale parameter Λ_{QCD}^2 determines much more weaker logarithmic deviation of the full gluon propagator from the free one in the $q^2 \rightarrow \infty$ limit (see Sect. 3.1 below and Refs. [3–6]). Then an interesting question arises within our approach. How does exactly the regularized mass gap provide the appearance of Λ_{QCD}^2 under the PT logarithms? The problem is that in the full gluon propagator the regularized mass gap contribution is linearly suppressed in comparison with the logarithmical divergent term in the PT $q^2 \rightarrow \infty$ limit (see Refs. [1, 2]).

4.1 Asymptotic Freedom

In the PT effective charge $d^{PT}(q^2) = [1 + \Pi^s(q^2; d^{PT})]^{-1}$ the invariant function $\Pi^s(q^2; d^{PT})$ can be only logarithmical divergent [1, 2] in the PT $q^2 \rightarrow \infty$ limit at any d , in particular at $d = d^{PT}$. So putting for further convenience $d^{PT}(q^2) = \alpha_s(q^2; \Lambda^2)/\alpha_s(\lambda)$ in this relation, one obtains

$$\alpha_s(q^2; \Lambda^2) = \frac{\alpha_s(\lambda)}{1 + b\alpha_s(\lambda)\ln(q^2/\Lambda^2)}, \tag{4.5}$$

and b is the standard color group factor. This expression represents the summation of the so-called main PT logarithms in powers of $\alpha_s(\lambda)$. However, nothing should depend on Λ (and hence on λ) when they go to infinity in order to recover the finite effective charge in this limit. To show explicitly that this finite limit exists, let us formally write

$$\Lambda^2 = f(\lambda)\Delta^2(\lambda, \alpha_s(\lambda)), \tag{4.6}$$

which is always valid, since $f(\lambda)$ is, in general, an arbitrary dimensionless function. In this connection, let us again remind that in order to get the expression (4.5) from the full effective charge $d(q^2) = [1 + \Pi^s(q^2; d) + c(d)(\Delta^2/q^2)]^{-1}$ [1, 2] the mass gap contribution Δ^2/q^2 is only asymptotically suppressed in the $q^2 \rightarrow \infty$ limit. In other words, we distinguish between the asymptotic suppression of the mass gap contribution Δ^2/q^2 and the formal PT $\Delta^2 = 0$ limit. So the mass gap $\Delta^2 = \Delta^2(\lambda, \alpha_s(\lambda))$ itself here is not put identically zero and hence the relation (4.6) makes sense. On account of the relation (4.4), it becomes

$$\Lambda^2 = f(\lambda)Z(\lambda, \alpha_s(\lambda))\Lambda_{QCD}^2, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow 0. \tag{4.7}$$

Substituting it into the expression (4.5) and doing some algebra, one obtains

$$\alpha_s(q^2) = \frac{\alpha_s}{1 + b\alpha_s \ln(q^2/\Lambda_{QCD}^2)}, \tag{4.8}$$

if and only if

$$\alpha_s = \frac{\alpha_s(\lambda)}{1 - b\alpha_s(\lambda) \ln(fZ)}, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow 0 \tag{4.9}$$

exists and is finite in the above shown limits. Here we introduce the short hand notations $f \equiv f(\lambda)$ and $Z \equiv Z(\lambda, \alpha_s(\lambda))$, for simplicity. Evidently, the finite α_s can be identified with the fine-structure constant of the strong interactions, calculated at some fixed scale. It is worth emphasizing that the existence and finiteness of α_s is due to the product (fZ) . Indeed, from (4.9) it follows

$$\ln(fZ) = \frac{\alpha_s - \alpha_s(\lambda)}{\alpha_s b\alpha_s(\lambda)} \rightarrow \frac{1}{b\alpha_s(\lambda)}, \quad \lambda \rightarrow \infty, \alpha_s(\lambda) \rightarrow 0, \tag{4.10}$$

which means that $(fZ) = \exp(1/b\alpha_s(\lambda))$ in the above shown limits. Substituting this into the relation (4.7) it becomes

$$\lim_{(\Lambda, \lambda) \rightarrow \infty} \Lambda^2 \exp\left(-\frac{1}{b\alpha_s(\lambda)}\right) = \Lambda_{QCD}^2, \quad \alpha_s(\lambda) \rightarrow 0, \tag{4.11}$$

which is the finite limit of the renormalization group equations solution [25].

At very large q^2 from (4.8) one recovers

$$\alpha_s(q^2) = \frac{1}{b \ln(q^2/\Lambda_{QCD}^2)}, \tag{4.12}$$

which is nothing but asymptotic freedom (AF) famous formula if $b > 0$ [3–6, 25]. In QCD with three colors and sixth flavors this is so, indeed. For the pure Yang-Mills (YM) fields $b = 11/4\pi > 0$ always. Let us underline that in the expressions (4.8) and (4.12) q^2 is always big enough, so it cannot go below Λ_{QCD}^2 . We have shown explicitly the AF behavior of QCD at

short distances ($q^2 \rightarrow \infty$), not using the renormalization group equations and their solutions (i.e., we need no expansion in powers of the coupling constant for the corresponding β -function) [3–6, 25]. The regularized mass gap is suppressed in the $q^2 \rightarrow \infty$ limit, as it has been mentioned above. From (4.4) and our consideration in this subsection it follows, nevertheless, that the regularized mass gap in the $\lambda \rightarrow \infty$ limit provides the existence of the asymptotic QCD scale parameter Λ_{QCD}^2 as well.

There is no relation between the renormalized mass gap Δ_R^2 (4.2) and the asymptotic scale parameter Λ_{QCD}^2 (4.4), since they show up explicitly at different regimes. They are different scales, indeed, responsible for different NP and non-trivial PT dynamics in QCD, though numerically they may not be very different, as underlined above. However, originally they have been generated in the region of small q^2 . In Ref. [25] it has been noticed that being numerically a few hundred MeV only, Λ_{QCD}^2 cannot survive in the $q^2 \rightarrow \infty$ limit. So none of the finite mass scale parameters can be determined by the PT QCD. They should come from the region of small q^2 , being thus NP by origin and surviving the renormalization program (i.e. the removal of λ in the $\lambda \rightarrow \infty$ limit), as was just demonstrated above.

Concluding, all this can be a manifestation that “the problems encountered in perturbation theory are not mere mathematical artifacts but rather signify deep properties of the full theory” [26]. The message that we are trying to convey is that the INP dynamical structure of the full gluon propagator indicates the existence of its nontrivial PT one and the other way around.

5 Confining Potential

After discussing some important aspects of the general MR program for the INP gluon propagator using the DRM and the DT, it makes sense to go back to the initial expansion for the INP gluon propagator (2.1). A new surprising feature of this solution is that its structure at zero ($q^2 \rightarrow 0$) can be again determined by its $\Delta^2/(q^2)^2$ term only. To show this explicitly, let us begin with the theorem from the theory of functions of complex variable [27], which is an extremely useful for the explanation of the behavior of our solution (2.1) in the deep IR $q^2 \rightarrow 0$ limit.

The function $L(q^2)$ is defined by its Laurent expansion (2.2), and thus it has an isolated essentially singular point at $q^2 = 0$. Its behavior in the neighborhood of this point is regulated by the Weierstrass-Sokhatsky-Casorati (WSC) theorem (see Appendix B) which tells that

$$\lim_{n \rightarrow \infty} L(q_n^2) = Y, \quad q_n^2 \rightarrow 0, \tag{5.1}$$

where Y is any complex number, and q_n^2 is a sequence of points $q_1^2, q_2^2, \dots, q_n^2, \dots$, along which q^2 goes to zero, and for which the above-displayed limit always exists. Of course, Y remains arbitrary (it depends on the chosen sequence of points q_n^2), but, in general, it depends on the same set of parameters as the residues, i.e., $Y \equiv Y(\lambda, \alpha, \xi, g^2)$. This theorem thus allows one to replace the Laurent expansion $L(q^2)$ by Y when $q^2 \rightarrow 0$ independently from all other test functions in the corresponding skeleton loop integrands, i.e.,

$$L(q^2) \rightarrow Y(\lambda, \alpha, \xi, g^2), \quad q^2 \rightarrow 0. \tag{5.2}$$

Our consideration in this section up to this point is necessarily formal, since the mass gap remains unrenormalized. So far it has been only regularized, i.e., $\Delta^2 \equiv \Delta^2(\lambda, \alpha, \xi, g^2)$. The

renormalization of the mass gap can be proceed as follows. Due to the above-formulated WSC theorem, the full gluon propagator (2.1) effectively becomes

$$D_{\mu\nu}(q) \equiv D_{\mu\nu}(q; \Delta^2) = iT_{\mu\nu}(q) \frac{1}{(q^2)^2} Y(\lambda, \alpha, \xi, g^2) \Delta^2(\lambda, \alpha, \xi, g^2), \quad q^2 \rightarrow 0, \quad (5.3)$$

so just the $\Delta^2(q^2)^{-2}$ structure of the full gluon propagator (2.1) is all that matters, indeed. Let us now define the renormalized (R) mass gap as follows:

$$\Delta_R^2 = Y(\lambda, \alpha, \xi, g^2) \Delta^2(\lambda, \alpha, \xi, g^2), \quad (5.4)$$

so that we consider $Y(\lambda, \alpha, \xi, g^2)$ as the MR constant for the mass gap, and Δ_R^2 is the physical mass gap within our approach. Precisely this quantity should be positive, finite, gauge-independent, etc., it should exist when $\lambda \rightarrow \infty$ and $\alpha \rightarrow 0$. Due to the WSC theorem, we can always choose such $Y = Y_1 \times Y_2 \times \dots$ in order to satisfy all the necessary requirements (each Y_n will depend on its own sequence of points along which $q^2 \rightarrow 0$, the so-called subsequences). Numerically it should be identified with the mass gap $\bar{\Delta}_R^2$ introduced by the relation (3.4) and described in Sect. 3 before Sect. 3.1 (see also the relation (5.8) below).

Thus the full gluon propagator relevant for the INP QCD dynamics becomes

$$D_{\mu\nu}(q; \Delta_R^2) = iT_{\mu\nu}(q) \frac{\Delta_R^2}{(q^2)^2}. \quad (5.5)$$

It is possible to say that because of the WSC theorem there always exists such a sequence of points along which $q^2 \rightarrow 0$ that the Laurent expansion (2.1)–(2.2) effectively convergence to the function (5.5) in the whole q^2 -momentum plane (containing both points $q^2 = 0$ and $q^2 = \infty$) after the renormalization of the mass gap is performed. But the region of small q^2 is of special interest. In this region the confinement dynamics begin to play a dominant role. However, it is worth emphasizing that the solution (5.5) is not the IR asymptotic ($q^2 \rightarrow 0$) of the initial Laurent expansion (2.1)–(2.2), since $(q^2)^{-2}$ term is not a leading one in this limit, it is a leading one just in the opposite $q^2 \rightarrow \infty$ limit.

The severe IR singularity $(q^2)^{-2}$, which only one is present in the relevant gluon propagator (5.5), is the first NP IR singularity possible in four-dimensional QCD. A special regularization expansion is to be used in order to deal with it, since the standard methods fail in this case. As mentioned above, it should be correctly treated within the DT [7] complemented by the DRM [8]. If q^2 is an independent skeleton loop variable, then the corresponding dimensional regularization of this singularity is given by the expansion (3.2) at $k = 0$, namely

$$(q^2)^{-2} = \frac{1}{\epsilon} \left[a(0) \delta^4(q) + O_0(\epsilon) \right], \quad \epsilon \rightarrow 0^+, \quad (5.6)$$

and $a(0) = \pi^2$. Due to the $\delta^4(q)$ function in the residue of this expansion, all the test functions which appear under corresponding skeleton loop integrals should be finally replaced by their expression at $q = 0$. So the dimensionally regularized expansion for the gluon propagator (5.5) becomes

$$D_{\mu\nu}(q; \Delta_R^2) = iT_{\mu\nu}(q) \Delta_R^2 \times \frac{1}{\epsilon} \left[a(0) \delta^4(q) + O(\epsilon) \right], \quad \epsilon \rightarrow 0^+, \quad (5.7)$$

where we put $O_0(\epsilon) = O(\epsilon)$, for simplicity. As emphasized above, in the presence of the IR regularization parameter ϵ all the Green’s functions and parameters become, in general,

dependent on it. The only way to remove the pole $1/\epsilon$ from the relevant gluon propagator (5.7) is to define the IR renormalized mass gap as follows:

$$\Delta_R^2 = X(\epsilon)\bar{\Delta}_R^2 = \epsilon \times \bar{\Delta}_R^2, \quad \epsilon \rightarrow 0^+, \tag{5.8}$$

where $X(\epsilon) = \epsilon$ is the IR MR constant for the mass gap. Let us remind that contrary to Δ_R^2 its IR renormalized counterpart $\bar{\Delta}_R^2$ exist as $\epsilon \rightarrow 0^+$. In both expressions for the mass gap the dependence on ϵ is assumed but not shown explicitly. So we distinguish between both mass gaps only by the dependence on ϵ .

On account of the relation (5.8), the dimensionally regularized expansion (5.7) finally becomes

$$D_{\mu\nu}(q; \bar{\Delta}_R^2) = iT_{\mu\nu}(q)\bar{\Delta}_R^2 a(0)\delta^4(q) + O(\epsilon), \quad \epsilon \rightarrow 0^+. \tag{5.9}$$

Evidently, after performing the renormalization program (i.e., going to the IR renormalized quantities), the terms of the order $O(\epsilon)$ can be omitted from the consideration.

The renormalization of the mass gap automatically IR renormalizes the relevant gluon propagator (5.5), so that

$$D_{\mu\nu}(q; \bar{\Delta}_R^2) = \epsilon \times iT_{\mu\nu}(q) \frac{\bar{\Delta}_R^2}{(q^2)^2}, \quad \epsilon \rightarrow 0^+, \tag{5.10}$$

in complete agreement with the expansion (3.9) at $k = 0$ with putting there $\bar{\Omega}_0^R = 1$. In order to achieve the agreement with the general renormalization program performed in Sect. 3 it is necessary to put $Y^{-1}(\lambda, \epsilon)X(\epsilon) = Z(\lambda, \epsilon)$ and hence $\Phi_0(\lambda; \epsilon) = \bar{\Omega}_0^R Y(\lambda, \epsilon) = Y(\lambda, \epsilon)$, on account of the convergence condition (3.6) at $k = 0$ and the relation (5.8).

There is no doubt left that the WSC theorem somehow underlines the importance of the simplest NP IR singularity $1/(q^2)^2$ possible in four-dimensional QCD, while all other may be suppressed in the deep IR ($q^2 \rightarrow 0$) region due to this theorem. However, from the general consideration in Sects. 3 and 4 the suppression mechanism is not seen, at least at this stage. In principle, one can develop the formal PT series in powers of the mass gap. Then the first nontrivial approximation is just the above-mentioned singularity $\Delta_R^2/(q^2)^2$, which is only one to be started with, anyway. It is well known that the expression (5.5) leads to the linear rising potential between heavy quarks [19] also “seen” by lattice QCD [20, 21]. That is why we call it the confining potential. Just it will be used for the derivation of the system of equations determining the confining quark propagator (for preliminary consideration see Ref. [28] and references therein). Evidently, what we have pointed out in Sect. 3.1 for the general case is valid for the confining potential (5.5) as well.

5.1 The Renormalized “Running” Effective Charge

It is instructive to find explicitly the corresponding β -function. From (5.5) it follows that the corresponding Lorentz structure, which is nothing but the corresponding effective charge (“running”), in terms of the renormalized mass gap is

$$d(q^2; \Delta_R^2) \equiv \alpha_s(q^2; \Delta_R^2) = \frac{\Delta_R^2}{q^2}, \tag{5.11}$$

and this does not depend whether the gluon momentum is independent loop variable or not. Then from the renormalization group equation for the renormalized effective charge, which

determines the β -function,

$$q^2 \frac{d\alpha_s(q^2; \Delta_R^2)}{dq^2} = \beta(\alpha_s(q^2; \Delta_R^2)), \quad (5.12)$$

it simply follows that

$$\beta(\alpha_s(q^2; \Delta_R^2)) = -\alpha_s(q^2; \Delta_R^2) = -\frac{\Delta_R^2}{q^2}. \quad (5.13)$$

Thus, one can conclude that the corresponding β -function as a function of its argument is always in the domain of attraction (i.e., negative). So it has no IR stable fixed point indeed as it is required for the confining theory [3]. Just these expressions for the β -function and the running effective charge should be used for the calculation of the truly NP quantities in the YM theory, such as the gluon condensate, the gluon part of the Bag constant [29–31], etc. In phenomenology for these purposes we need to know the ratio $\beta(q^2)/\alpha_s(q^2)$ in the $q^2 \rightarrow 0$ limit, so it is always

$$\frac{\beta(q^2)}{\alpha_s(q^2)} \equiv \frac{\beta(\alpha_s(q^2; \Delta_R^2))}{\alpha_s(q^2; \Delta_R^2)} = -1 \quad (5.14)$$

within our approach to low-energy QCD, that's INP QCD (don't forget that the replacement $\alpha_s(q^2) \rightarrow \alpha_s^{INP}(q^2)$ and $\beta(q^2) \rightarrow \beta^{INP}(q^2)$ is assumed [2]).

Concluding, let us note if the renormalized effective charge (5.11) is to be multiplied by the additional powers of $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, \dots$, then this product should be treated as the full gluon propagator itself, described above. The only difference between them becomes the unimportant tensor structure $T_{\mu\nu}(q)$.

6 Discussion

Thus the QCD vacuum is really beset with severe IR singularities, which have been summarized (accumulated) into the INP gluon propagator (2.1). There is no doubt that the purely transversal severely singular virtual gluon field configurations play important role in the dynamical and topological structure of the QCD ground state, leading thus to the general zero momentum modes enhancement (ZMME) effect there reflected in the INP gluon propagator. Evidently, the ZMME (or simply ZME) mechanism of confinement (see our previous papers [32, 33] and references therein as well) is nothing but the well forgotten IR slavery (IRS) one, which can be equivalently referred to as a strong coupling regime [3, 16].

Indeed, at the very beginning of QCD it was expressed a general idea [16, 34–40] that the quantum excitations of the IR degrees of freedom, because of self-interaction of massless gluons in the QCD vacuum, made it only possible to understand confinement, dynamical (spontaneous) breakdown of chiral symmetry and other NP effects. In other words, the importance of the deep IR structure of the QCD vacuum has been emphasized as well as its relevance to the above-mentioned NP effects and the other way around. This development was stopped by the wide-spread wrong opinion that severe IR singularities cannot be put under control. Here we have explicitly shown (see also our recent papers [9, 41] and references therein) that the adequate mathematical theory for quantum YM gauge theory is the DT (the theory of generalized functions) [7], complemented by the DRM [8]. Together with the theory of functions of complex variable [27] they provide a correct treatment of these

severe IR singularities without any problems. Thus, we come back to the old idea but on a new basis that is why it becomes new (“new is well forgotten old”). In other words, we put the IRS mechanism of color confinement on a firm mathematical ground. This makes it possible to analytically formulate the gluon confinement criterion in a gauge invariant way for the first time.

The confining potential (5.5) in the different approximations and gauges has been earlier obtained and investigated in many papers (see, for example, Ref. [42] and references therein). We have confirmed and thus revitalized these investigations, in which this behavior has been obtained as an IR asymptotic solution to the gluon SD equation. However, let us emphasize once more that due to the WSC theorem the confining potential (5.5) is not the IR asymptotic of the initial Laurent expansion (2.1)–(2.2). Moreover, the whole INP gluon propagator (2.1) exactly converges to the confining potential (5.5) after the renormalization of the mass gap is performed. The WSC theorem clearly shows that the $\Delta_R^2/(q^2)^2$ structure is only important, while all other terms in the INP gluon propagator (2.1) are suppressed (though each next term in the expansion (2.2) is more singular in the IR than the previous one).

In the presence of the mass gap the coupling constant plays no role. This is also an evidence of the “dimensional transmutation”, $g^2 \rightarrow \Delta^2(\lambda, \alpha, \xi, g^2)$ [3, 43, 44], which occurs whenever a massless theory acquires masses dynamically. It is a general feature of spontaneous symmetry breaking in field theories. In our case, the color gauge symmetry is broken at the level of the full gluon self-energy, while maintaining at the level of the full gluon propagator, which is more important [1] (something like a “self-consistent violation” or a “hidden violation” of the color gauge symmetry). In the massive solution [2] the mass gap transforms further into the effective gluon mass, i.e., $g^2 \rightarrow \Delta^2(\lambda, \alpha, \xi, g^2) \rightarrow m_g(\xi)$. Nevertheless, it remains gauge-dependent (i.e., not physical) even after the corresponding full gluon propagator is renormalized. Within the INP QCD the mass gap transforms further into the physical mass gap, i.e., $g^2 \rightarrow \Delta^2(\lambda, \alpha, \xi, g^2) \rightarrow \Delta_R^2$, and the gluons remain massless in a gauge invariant way (this paper). Let us note that the renormalization of the mass gap automatically renormalizes the INP QCD gluon propagator as well.

7 Conclusions

Let us denote the version of our mass gap $\bar{\Delta}_R^2$ which will appear in the S -matrix elements for the corresponding physical quantities/processes in low-energy QCD as Λ_{INP}^2 (in principle, they may be slightly different from each other, indeed). Then a symbolic relation between it and the initial mass gap $\Delta^2(\lambda, \alpha_s(\lambda))$ and Λ_{PT}^2 instead of Λ_{QCD}^2 (for reason see discussion below) could be written as follows:

$$\Lambda_{INP}^2 \xleftarrow[\infty \leftarrow \lambda]{\infty \leftarrow \alpha_s(\lambda)} \Delta^2(\lambda, \alpha_s(\lambda)) \xrightarrow[\lambda \rightarrow \infty]{\alpha_s(\lambda) \rightarrow 0} \Lambda_{PT}^2, \tag{7.1}$$

which summarizes our main results in this investigation. QCD as a quantum gauge field theory, describing the interactions of never seen colored objects (the gluons and quarks), cannot have the physical mass gap. In other words, this is a theory which describes the behavior of the colored objects in the vacuum. In QCD mass gap may only appear in the way described in our previous publication [1], that’s $\Delta^2(\lambda, \alpha_s(\lambda))$ in (7.1). In order to become the theory of the strong interactions it should undergo the two phase transitions; in the weak and strong coupling regimes. In the first case it becomes the PT QCD which describes all the high-energy phenomena in the strong interactions from first principles (AF, scale violation,

hard processes, etc.). It has its own physical mass gap which we denote as Λ_{PT}^2 in (7.1). In the second case it becomes the INP QCD which describes all the low-energy phenomena in the strong interactions from first principles (those includes first of all color confinement, dynamical breakdown of chiral symmetry, bound-states, etc.). It has its own physical mass gap, that's Λ_{INP}^2 in (7.1).

In this connection, a few things should be made perfectly clear. First of all, let us underline that the PT QCD and the INP QCD are not effective theories, as pointed out above both theories are fundamental ones. Secondly, such a quantity as Λ_{QCD}^2 does not exist at all, since QCD itself cannot have a physical limit. In order to avoid any confusion the corresponding scale is better to denote as Λ_{PT}^2 , since just the PT QCD is responsible for all the high-energy phenomena in the strong interactions. Thus similarly the relation (7.1), the following symbolic relation makes sense

$$INP\ QCD \iff QCD \implies PT\ QCD, \quad (7.2)$$

so that at the fundamental (quark-gluon) level the PT QCD is AF, while the INP QCD confines gluons. In the subsequent papers we will show that this theory will confine quarks, as well as will explain the spontaneous breakdown of chiral symmetry. Both theories have their own mass gaps Λ_{INP}^2 and Λ_{PT}^2 , which are solely responsible for the large- and short-scale structures of the QCD ground state, respectively.

A few years ago Jaffe and Witten have formulated the following theorem [24]:

Yang–Mills Existence And Mass Gap: Prove that for any compact simple gauge group G , quantum Yang–Mills theory on \mathbf{R}^4 exists and has a mass gap $\Delta > 0$.

Of course, to prove the existence of the YM theory with compact simple gauge group G is a formidable task yet. It is rather a mathematical than a physical problem. However, from the JW presentation of their theorem it clearly follows that their mass gap should be identified with our mass gap Λ_{INP}^2 . At the same time, we have argued above that QCD itself cannot have a physical mass gap. It has a mass gap which is only regularized, i.e., $\Delta^2 \equiv \Delta^2(\lambda, \alpha_s(\lambda))$, and therefore there is no guarantee that it is positive. It cannot be related directly to any of physical quantities/processes. Let us also remind that QCD cannot confine free gluons [1, 2]. As actual theory of the strong interactions the two different faces of QCD come into the play: the PT QCD for high-energy physics and INP QCD for low-energy physics, which confines “dressed” gluons, while free gluons do not exist in this theory. The corresponding mass gaps have now physical meanings (they are finite, positive, gauge-invariant, etc.).

Our basic result obtained in the previous works [1, 2] and in this paper can be jointly formulated as follows:

Mass Gap Existence And Gluon Confinement: If quantum Yang–Mills theory with compact simple gauge group $G = SU(3)$ exists on \mathbf{R}^4 , then undergoing the phase transition in the strong coupling regime it becomes INP QCD, which has a physical mass gap and confines gluons.

Some important features of the INP QCD are:

1. Its full gluon propagator (2.1) converges to the expression (5.5) after the renormalization of the mass gap is performed. This expression is effectively valid in the whole q^2 -momentum plane.
2. It has a physical mass gap.
3. It confines “dressed” gluons in asymptotic states.

4. It has no free gluons at all due to the subtraction method proposed and formulated in our previous works [1, 2].

Some other interesting features of this theory may be established after the explicit including of the quark degrees of freedom into its formalism in the next our papers. However, first in the subsequent paper we will show how the QCD vacuum structure is to be investigated within the YM version of the INP QCD.

Concluding, a few remarks are in order. In our previous work [1], we have explicitly shown that the full photon propagator in quantum electrodynamics (QED) has only the PT-type IR singularity, $1/q^2$. This is in agreement with the cluster property of the Wightman functions [45], that's correlation functions of observables. In QCD the explicit presence of the regularized mass gap, which are necessarily accompanied by severe IR singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, \dots$, apparently, will violate this property. In turn, this validates the Strocchi theorem [46], which allows such a severely singular behavior of the full gluon propagator in QCD. However, this is not a problem, since QCD has no physical observables. In PT QCD, which gluon propagator is as much singular as $1/q^2$ only, the cluster property will not be violated. On the other hand, in INP QCD with a such singular behavior of the relevant gluon propagator (5.5) the situation with the Wightman functions is not clear. It can be clarified only after the solution of the color confinement problem, and a realistic calculations of the various physical observables within INP QCD. At the fundamental quark-gluon level only these remarks make sense about the correlation between the structure of the corresponding gluon propagator and the properties of the Wightman functions.

Acknowledgements Support by HAS-JINR grant (P. Levai) is to be acknowledged. The author is grateful to P. Forgács, J. Nyíri, T. Biró, C. Wilkin, A. Lukács, M. Vasúht, G. Barnaföldi and especially to A.V. Kouzushin for useful discussions, remarks and help.

Appendix A: Infrared Dimensional Regularization Within the Distribution Theory

8.1 The DRM in the PT

As repeatedly emphasized in our previous works [1, 2], the mass gap in nothing but the re-defined skeleton tadpole term, or it can be reduced to the skeleton tadpole term itself. Also, it has been explicitly shown that there is no such regularization scheme (preserving or not gauge invariance) in which the transversality condition for the full gluon self-energy could be satisfied unless the constant skeleton tadpole term

$$\Pi'_{\rho\sigma}(D) \equiv \Pi_t(D) \equiv \Delta_t^2(D) = g^2 \int \frac{id^4q_1}{(2\pi)^4} T_4^0 D(q_1), \quad (8.1)$$

is to be disregarded from the very beginning, i.e., put formally zero everywhere. Here T_4^0 is the four-gluon point-like vertex, and g^2 is the dimensionless coupling constant squared. We omit the tensor and color indices in this integral, as unimportant for further discussion. It is nothing else but the quadratically divergent in the PT constant, so it is assumed to be regularized. The mass gap is not survived in the PT $q^2 \rightarrow \infty$ limit [1], however, in the PT there are still problems with such kind of integrals.

In the PT, when the first non-trivial approximation for the full gluon propagator $D = D(q)$ is the free one $D_0 = D_0(q)$, the constant tadpole term is to be simply discarded, i.e., to be put formally zero within the DRM [8], so that $\Pi'_{\rho\sigma}(D_0) = \delta_{\rho\sigma} \Delta_t^2(D_0) = 0$. However, even in the DRM this is not an exact result, but rather an embarrassing prescription,

as pointed out in Ref. [5]. To show explicitly that there are still problems, as mentioned above, it is instructive to substitute the first iteration of the gluon SD equation into the previous expression (A1). Symbolically it looks like $D(q) = D_0(q) + D_0(q)i\Pi(q; D)D(q) = D_0(q) + D_0(q)i\Pi(q; D_0)D_0(q) + \dots = D_0(q) + D^{(1)}(q) + \dots$, where we omit all the indices and put $D_0 \equiv D^{(0)}$. Doing so, one obtains

$$\begin{aligned} \Pi_l(D = D_0 + D^{(1)} + \dots) &= \Pi_l(D_0) + \Pi_l(D^{(1)}) + \dots \\ &= \Pi_l(D_0) + g^2 \int \frac{id^4q_1}{(2\pi)^4} T_4^0[D_0(q_1)]^2 i\Pi(q_1; D_0) + \dots \\ &= \Pi_l(D_0) + \Pi_l(D_0)g^2 \int \frac{i^2d^4q_1}{(2\pi)^4} T_4^0[D_0(q_1)]^2 \\ &\quad + g^2 \int \frac{i^2d^4q_1}{(2\pi)^4} T_4^0[D_0(q_1)]^2 \Pi^s(q_1; D_0) + \dots \end{aligned} \tag{8.2}$$

Here we introduce the subtraction as follows: $\Pi^s(q_1; D_0) = \Pi(q_1; D_0) - \Pi(0; D_0)$, and put $\Pi(0; D_0) = \Pi_l(D_0)$, for simplicity, when the mass gap is to be reduced to the tadpole term itself [1]. In the second line of (8.2) the first integral is not only UV divergent but IR singular as well. If we now omit the first term in accordance with the above-mentioned prescription, the product of this integral and the tadpole term $\Pi_l(D_0)$ remains, nevertheless, undetermined. Moreover, the structure of the second integral in this line is much more complicated than in the divergent constant integral $\Pi_l(D_0)$ in (8.1). All this reflects the general problem that such kind of massless integrals

$$\int \frac{d^d q}{(2\pi)^d} \frac{q_{\mu_1} \dots q_{\mu_p}}{(q^2)^n} \tag{8.3}$$

are ill defined, since there is no dimension where they are meaningful. They are either IR singular or UV divergent, depending on the relation between the numbers d , p and n [5]. This prescription clearly shows that the DRM, preserving gauge invariance, nevertheless, is by itself not sufficient to provide us insights into the correct treatment of the power-type IR singularities shown in (8.3) (we will address this problem below in Sect. 2). Thus, one concludes that the tadpole term (A1) $\Delta_l^2(D) \equiv \Delta_l^2(\lambda, \alpha; D)$ is, in general, not zero.

However, in the PT we can adhere to the prescription that such massless tadpole integrals can be discarded in the DRM [5, 8]. As mentioned above, we have already shown [1] that the mass gap, in general, and the skeleton tadpole term, in particular, can be neglected in the PT, indeed (not depending on whether λ, α are to be introduced within the regularization scheme preserving gauge invariance or not). In what follows we will show how precisely the DRM [8] should be correctly implemented into the DT [7] in order to control the power-type severe IR singularities, which may appear not only in the PT series, but mainly in the NP QCD as well (see Ref. [2]).

8.2 The DRM in the DT

In general, all the Green’s functions in QCD are generalized functions, i.e., they are distributions. This is especially true for the NP IR singularities of the full gluon propagator due to the self-interaction of massless gluons in the QCD vacuum. They present a rather broad and important class of functions with algebraic singularities, i.e., functions with nonsummable singularities at isolated points [7] (at zero in our case). Roughly speaking, this means that all

relations involving distributions should be considered under corresponding integrals, taking into account the smoothness properties of the corresponding space of test functions. Let us note in advance that the space in which our generalized functions are continuous linear functionals is K , that's the space of infinitely differentiable functions having compact support, i.e., they are zero outside some finite region (different for each differentiable function) [7].

Let us consider the positively definite ($P > 0$) squared (quadratic) Euclidean form

$$P(q) = q_0^2 + q_1^2 + q_2^2 + \dots + q_{n-1}^2 = q^2, \tag{8.4}$$

where n is the number of the components. The generalized function (distribution) $P^\lambda(q)$, where λ being, in general, an arbitrary complex number, is defined as

$$(P^\lambda, \varphi) = \int_{P>0} P^\lambda(q)\varphi(q)d^d q, \tag{8.5}$$

where $\varphi(q)$ is the above-mentioned some test function. At $Re\lambda \geq 0$ this integral is convergent and is an analytic function of λ . Analytical continuation to the region $Re\lambda < 0$ shows that it has a simple pole at points [7]

$$\lambda = -\frac{n}{2} - k, \quad k = 0, 1, 2, 3, \dots \tag{8.6}$$

In order to actually define the skeleton loop integrals in the deep IR domain, which appear in the system of the SD equations, it is necessary to introduce the IR regularization parameter ϵ , defined as $d = n + 2\epsilon$, $\epsilon \rightarrow 0^+$ within the DRM [8], where d is the dimension of the loop integral (see (A5)). As a result, all the Green's functions and "bare" parameters should be regularized with respect to ϵ which should be set to zero at the end of the computations. The structure of the NP IR singularities is then determined (when n is even number) as follows [7]:

$$(q^2)^\lambda = \frac{C_{-1}^{(k)}}{\lambda + (d/2) + k} + \text{finite terms}, \tag{8.7}$$

where the residue is

$$C_{-1}^{(k)} = \frac{\pi^{n/2}}{2^{2k} k! \Gamma((n/2) + k)} \times L^k \delta^n(q) \tag{8.8}$$

with $L = (\partial^2/\partial q_0^2) + (\partial^2/\partial q_1^2) + \dots + (\partial^2/\partial q_{n-1}^2)$.

Thus the regularization of the NP IR singularities (8.5), on account of (8.6), is nothing but the whole expansion in the corresponding powers of ϵ and not the separate term(s). Let us underline its most remarkable feature. The order of singularity does not depend on λ , n and k . In terms of the IR regularization parameter ϵ it is always a simple pole $1/\epsilon$. This means that all power terms in (8.7) will have the same singularity, i.e.,

$$(q^2)^{-\frac{n}{2}-k} = \frac{1}{\epsilon} C_{-1}^{(k)} + \text{finite terms}, \quad \epsilon \rightarrow 0^+, \tag{8.9}$$

where we can put $d = n$ now (i.e., after introducing this expansion). By "finite terms" here and everywhere a number of necessary subtractions under corresponding integrals is understood [7]. However, the residue at a pole will be drastically changed from one power singularity to another. This means different solutions to the whole system of the SD equations for different set of numbers λ and k . Different solutions mean, in their turn, different

vacua. In this picture different vacua are to be labeled by the two independent numbers: the exponent λ and k . At a given number of $d(=n)$ the exponent λ is always negative being integer if $d(=n)$ is an even number or fractional if $d(=n)$ is an odd number. The number k is always integer and positive and precisely it determines the corresponding residue at a simple pole, see (8.9). It would not be surprising if these numbers were somehow related to the nontrivial topology of the QCD vacuum in any dimensions.

It is worth emphasizing that the structure of severe IR singularities in Euclidean space is much simpler than in Minkowski space, where kinematical (unphysical) singularities due to the light cone also exist [3, 7, 47] (in this connection let us remind that in Euclidean metrics $q^2 = 0$ implies $q_i = 0$ and vice-versa, while in Minkowski metrics this is not so). In this case it is rather difficult to untangle them correctly from the dynamical singularities, the only ones which are important for the calculation of any physical observable. Also, the consideration is much more complicated in the configuration space [7]. That is why we always prefer to work in the momentum space (where propagators do not depend explicitly on the number of dimensions) with Euclidean signature. We also prefer to work in the covariant gauges in order to avoid peculiarities of the non-covariant gauges [3, 48, 49], for example, how to untangle the gauge pole from the dynamical one.

In principle, none of the regularization schemes (how to introduce the IR regularization parameter in order to parameterize the NP IR divergences and thus to put them under control) should be introduced by hand. First of all, it should be well defined. Secondly, it should be compatible with the DT [7]. The DRM [8] is well defined, and here we have shown how it should be introduced into the DT (complemented by the number of subtractions, if necessary). Though the so-called $\pm i\epsilon$ regularization is formally equivalent to the regularization used in our paper (see again Ref. [7]), nevertheless, it is rather inconvenient for practical use. Especially this is true for the gauge-field propagators, which are substantially modified due to the response of the vacuum (the $\pm i\epsilon$ prescription is designated for and is applicable only to the theories with the PT vacua, indeed [10, 50]). Other regularization schemes are also available, for example, such as analytical regularization used in Ref. [16] or the so-called Speer's regularization [51]. However, they should be compatible with the DT as emphasized above. Anyway, not the regularization is important but the DT itself. Just this theory provides an adequate mathematical framework for the correct treatment of all the Green's functions in QCD (apparently, for the first time the distribution nature of the Green's functions in quantum field theory has been recognized and used in Ref. [52]).

The regularization of the NP IR singularities in QCD is determined by the Laurent expansion (8.9) at $n = 4$ as follows:

$$(q^2)^{-2-k} = \frac{1}{\epsilon} a(k) [\delta^4(q)]^{(k)} + f.t. = \frac{1}{\epsilon} \left[a(k) [\delta^4(q)]^{(k)} + O_k(\epsilon) \right], \quad \epsilon \rightarrow 0^+, \quad (8.10)$$

where $a(k) = \pi^2/2^{2k} k! \Gamma(2+k)$ is a finite constant depending only on k and $[\delta^4(q)]^{(k)}$ represents the k 's derivative of the δ -function (see (8.7) and (8.8)). We point out that after introducing this expansion everywhere one can fix the number of dimensions, i.e., put $d = n = 4$ for QCD without any further problems. Indeed there will be no other severe IR singularities with respect to ϵ as it goes to zero, but those explicitly shown in this expansion. Let us underline that, while the initial expansion (2.1) is the Laurent expansion in the inverse powers of the gluon momentum squared, the regularization expansion (8.10) is the Laurent expansion in powers of ϵ . This means that its regular part is as follows: $f.t. = (q^2)^{-2-k} + \epsilon (q^2)^{-2-k} \ln q^2 + O(\epsilon^2)$, where for the unimportant here definition of the functional $(q^2)^{-2-k}$ see Ref. [7]. These terms, however, play no any role in the IRMR program which has been discussed in Sect. 3. The dimensionally regularized expansion (A10)

takes place only in four-dimensional QCD with Euclidean signature. In other dimensions and/or Minkowski signature it is much more complicated as pointed out above. As it follows from this expansion any power-type NP IR singularity, including the simplest one at $k = 0$, scales as $1/\epsilon$ as it goes to zero. Just this plays a crucial role in the IR renormalization of the theory within our approach. Evidently, such kind of the dimensionally regularized expansion (8.10) does not exist for the PT IR singularity, which is as much singular as $(q^2)^{-1}$ only.

In summary, first we have emphasized the distribution nature of the NP IR singularities. Secondly, we have explicitly shown how the DRM should be correctly and in a gauge invariant way implemented into the DT. This makes it possible to put severe IR singularities under firm mathematical control.

Appendix B: The WSC Theorem

One of the main theorems in the theory of functions of complex variable [27] is the above-mentioned WSC theorem. It describes the behavior of meromorphic functions near essential singularities.

Theorem (Weierstrass–Sokhatsky–Casorati) *If z_0 is an essential singularity of the function $f(z)$, then for any complex number Z there exists the sequence of points $z_k \rightarrow z_0$, such that*

$$\lim_{k \rightarrow \infty} f(z_k) = Z. \tag{B.1}$$

So this theorem tells us that the behavior of the function $f(z)$ near its essential singularity z_0 is not determined, i.e., in fact, it remains arbitrary. It depends on the chosen sequence of points z_k along which z goes to zero, that's $Z \equiv Z(z_k)$ (do not mixed this complex number with the MR constant of the mass gap in the relation (3.4)).

Let us consider one classical example [27]. The function

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} \tag{B.2}$$

has the above shown Laurent series about the essential singularity at $z = 0$, i.e., this Laurent expansion converges to the function $f(z)$ everywhere apart from the point $z = 0$. At the same time, due to the WSC theorem the behavior of the function $f(z)$ near $z = 0$, and hence of the Laurent expansion itself, depends on the chosen sequence of points z_k along which $z \rightarrow 0$. So let us proceed as follows:

(i). If one chooses $z_k = 1/k, k = 1, 2, 3, \dots$, then

$$\lim_{k \rightarrow \infty} f(z_k) = \lim_{k \rightarrow \infty} e^k = \infty. \tag{B.3}$$

(ii). If one chooses $z_k = -1/k, k = 1, 2, 3, \dots$, then

$$\lim_{k \rightarrow \infty} f(z_k) = \lim_{k \rightarrow \infty} e^{-k} = 0. \tag{B.4}$$

(iii). If one chooses $z_k = 1/\ln A + 2k\pi i, k = 0, 1, 2, 3, \dots$, then

$$\lim_{k \rightarrow \infty} f(z_k) = \lim_{k \rightarrow \infty} e^{\ln A + 2k\pi i} = A, \tag{B.5}$$

where A is some finite constant. Many other examples can be found in Ref. [27] and in other text-books on the theory of functions of complex variable.

Concluding, let us make one important point perfectly clear. For example, the function

$$f(z) = \frac{z}{\sqrt{1+z^2}} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n} (n!)^2} \frac{1}{z^{2n}} \quad (\text{B.6})$$

has no singularity at zero at all, while its Laurent expansion always has an essential singularity at $z = 0$, by definition. The equality (B.6) means that this Laurent expansion converges to the above shown function $f(z)$ in the ring which excludes zero point. Its region of convergence is $1 < |z| < \infty$, while the behavior of any Laurent expansion near its essential singularity is always governed by the WSC theorem, in particular of the Laurent expansion shown in the right-hand-side of (B.6).

Another characteristic example is the function

$$f(z) = \frac{z}{z-1} = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n, \quad (\text{B.7})$$

which is nothing but geometric series. The Laurent expansion, shown in the right-hand-side of this equation, converges to the function $f(z)$ also in the region $1 < |z| < \infty$, while the behavior of the Laurent expansion itself at $z \rightarrow 0$ is again governed by the WSC theorem. The same is true for the Laurent expansion in (B.2). It converges to the function $\exp(1/z)$ in the whole complex plane apart from the point $z = 0$, while near this point its behavior is uncertain, as it was described above. The message we are trying to convey is that all the equalities containing the Laurent expansions should be treated carefully in accordance with the above-mentioned theorem (i.e., the region of convergence should be fixed clearly, otherwise the equality can be incorrectly understood).

References

- Gogokhia, V.: Int. J. Theor. Phys. (2009). doi:[10.1007/s10773-009-0101-3](https://doi.org/10.1007/s10773-009-0101-3); [arXiv:0806.0247](https://arxiv.org/abs/0806.0247) [hep-th] [hep-ph]
- Gogokhia, V.: Int. J. Theor. Phys. (2009). doi:[10.1007/s10773-009-0151-6](https://doi.org/10.1007/s10773-009-0151-6); [arXiv:0904.2266](https://arxiv.org/abs/0904.2266) [hep-ph] [hep-th]
- Marciano, W., Pagels, H.: Phys. Rep. C **36**, 137 (1978)
- Peskin, M.E., Schroeder, D.V.: An Introduction to Quantum Field Theory. AW, Advanced Book Program (1995)
- Itzykson, C., Zuber, J.-B.: Quantum Field Theory. McGraw-Hill, New York (1984)
- Muta, T.: Foundations of QCD. World Scientific, Singapore (1987)
- Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. I, II. Academic Press, New York (1968)
- 't Hooft, G., Veltman, M.: Nucl. Phys. B **44**, 189 (1972)
- Gogohia, V.: Phys. Lett. B **584**, 225 (2004)
- Gribov, V.N.: Gauge Theories and Quark Confinement. PHASIS, Moscow (2002)
- Preparata, G.: Phys. Rev. D **7**, 2973 (1973)
- Gogohia, V., Kluge, G., Prisznyak, M.: [arXiv:hep-ph/9509427](https://arxiv.org/abs/hep-ph/9509427)
- Pocsik, G., Torma, T.: Acta Phys. Hung. **62**, 101, 107 (1987)
- Fodor, Z.: Acta Phys. Polonica B **19**, 21 (1988)
- 't Hooft, G.: Nucl. Phys. B **75**, 461 (1974)
- Pagels, H.: Phys. Rev. D **15**, 2991 (1977)
- Wilson, K.B.: Phys. Rev. D **10**, 2445 (1974)
- Bander, M.: Phys. Rep. **75**, 205 (1981)
- Alekseev, A.I., Arbuzov, B.A.: Phys. Atom. Nucl. **61**, 264 (1998)

20. Born, K.D., et al.: Phys. Lett. B **329**, 325 (1994)
21. Miller, V.M., et al.: Phys. Lett. B **335**, 71 (1994)
22. Quark Matter 2005, Edited by T. Csorgo, G. David, P. Levai, G. Papp (Elsevier, Amsterdam, 2005)
23. Gyulassy, M., McLerran, L.: [arXiv:nucl-th/0405013](https://arxiv.org/abs/nucl-th/0405013)
24. Jaffe, A., Witten, E.: Yang–Mills existence and mass gap. <http://www.claymath.org/prize-problems/>, <http://www.arthurjaffe.com>
25. Andreev, I.V.: QCD and Hard Processes at High Energies. Nauka, Moscow (1981). Russ. Ed.
26. Wilczek, F., Proc. Inter Conf., QCD–20 Years Later, v. 1. Aachen, June 9–13 (1992)
27. Lavrentiev, M.A., Shabat, B.V.: Methods of the Theory of Functions of Complex Variable, Nauka, Moscow (1987). (Russ. Ed.)
28. Gogokhia, V.: [arXiv:0704.3189](https://arxiv.org/abs/0704.3189) [hep-ph]
29. Gogohia, V., Kluge, G.: Phys. Rev. D **62**, 076008 (2000)
30. Gogohia, V., et al.: Int. J. Mod. Phys. A **15**, 45 (2000)
31. Barnaföldi, G.G., Gogokhia, V.: [arXiv:0708.0163](https://arxiv.org/abs/0708.0163) [hep-ph]
32. Gogohia, V.: Phys. Lett. B **485**, 162 (2000)
33. Gogohia, V., et al.: Phys. Lett. B **453**, 281 (1999)
34. Susskind, L., Kogut, J.: Phys. Rep. C **23**, 348 (1976)
35. Fritzsch, H., Gell-Mann, M., Leutwyler, H.: Phys. Lett. B **47**, 365 (1973)
36. Weinberg, S.: Phys. Rev. Lett. **31**, 494 (1973)
37. Georgi, H., Glashow, S.: Phys. Rev. Lett. **32**, 438 (1974)
38. Gervais, J.L., Neveu, A.: Phys. Rep. C **23**, 240 (1976)
39. Mandelstam, S.: Phys. Rev. D **20**, 3223 (1979)
40. Matinyan, S.G., Savvidy, G.K.: Nucl. Phys. B **134**, 539 (1978)
41. Gogohia, V.: Phys. Lett. B **618**, 103 (2005)
42. Gogohia, V.: [arXiv:hep-ph/0311061](https://arxiv.org/abs/hep-ph/0311061)
43. Coleman, S., Weinberg, E.: Phys. Rev. D **7**, 1888 (1973)
44. Gross, D.J., Neveu, A.: Phys. Rev. D **10**, 3235 (1974)
45. Streater, R., Wightman, A.: Spin and Statistics and All That. Benjamin, New York (1964)
46. Strocchi, F.: Phys. Lett. B **62**, 60 (1976)
47. Ball, J.S., Chiu, T.-W.: Phys. Rev. D **22**, 2542, 2550 (1980)
48. Leibbrandt, G.: Noncovariant Gauges. World Scientific, Singapore (1994)
49. Bassetto, A., Nardelli, G., Soldati, R.: Yang–Mills Theories in Algebraic Non Covariant Gauges. World Scientific, Singapore (1991)
50. Dokshitzer, Y.L., Kharzeev, D.E.: [arXiv:hep-ph/0404216](https://arxiv.org/abs/hep-ph/0404216)
51. Speer, E.R.: J. Math. Phys. **15**, 1 (1974)
52. Bogoliubov, N.N., Shirkov, D.V.: Introduction to the Theory of Quantised Fields. Interscience, New York (1959)